It is clear, from (20), that for this case the multiplication number required is $[Q+(Q+1)/2]\ (L-1)/2$ instead of $Q(L-1)$.

From the above discussions we see that, by suitably arranging arithmetic operations, the number of multiplications required can be reduced by about 25 percent. In case that a serial/parallel multiplier (such as Am25LS14 manufactured by Advanced Micro Devices, Inc.) is used for the filter implementation, a 25 percent reduction on the multiplication operations means a 25 percent increase on the filter speed because almost all nonmultiplication operations can be performed during the first half multiplication cycle of the multiplier while unwanted less significant bits are being generated.

IV. Conclusions

In this paper we examine some problems on the implementation of digital interpolator using linear-phase FIR filters. A procedure for selecting parameters L, N, and Q is presented. It is shown that, except for one interpolated sample in *Case A-2* where L is even and Q is odd, all $(L-1)$ interpolated samples can be computed from the same set of Q original input samples. This fact can greatly simplify the design of the control section of interpolation filters.

The symmetry property of the impulse response of linear-phase FIR filters is exploited. It is shown that, by suitably arranging arithmetic operations, the number of multiplication operations required can be reduced about 25 percent. If a serial/parallel multiplier is used for the filter implementation, a 25 percent reduction on the multiplication operations means a 25 percent increase on the filter speed.

References

[1] R. W. Schafer and L. R. Rabiner, "A digital signal processing approach to interpolation," *Proc. IEEE*, vol. 61, pp. 692–702, June 1973.
[2] G. Oetken, T. W. Parks, and H. W. Schussler, "New results in the design of digital interpolators," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-23, pp. 301–309, June 1975.
[3] R. E. Crochiere and L. R. Rabiner, "Optimum FIR digital filter implementations for decimation, interpolation, and narrow-band filtering," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-23, pp. 444–456, Oct. 1975.
[4] R. R. Shively, "On multistage FIR filters with decimation," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-23, pp. 353–357, Aug. 1975.
[5] O. Hermann, L. R. Rabiner, and D. S. K. Chang, "Practical design rules for optimal FIR lowpass digital filters," *Bell Syst. Tech. J.*, vol. 52, pp. 769–800, July-Aug. 1973.
[6] J. H. McClellan, T. W. Parks, and L. R. Rabiner, "A computer program for designing optimum FIR linear phase digital filters," *IEEE Trans. Audio Electroacoust.*, vol. AU-21, pp. 506–526, Dec. 1973.

Transformed Coherence Functions for Multivariate Studies

ALAN G. JONES

Abstract—**In this paper, transformed coherence function estimates are defined which display several desirable properties when compared with the conventional forms; 1) their probability distribution functions are more nearly normal, 2) their mean values are normalized to a value of unity for totally uncorrelated data, and 3) their variances are independent of the true values.**

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I. Introduction

The magnitude-squared coherence function (MSC) between two jointly stationary random processes, $x(t)$ and $y(t)$, is defined as

$$|\gamma(f)|^2 = \frac{|G_{xy}(f)|^2}{G_{xx}(f)\,G_{yy}(f)} \tag{1}$$

where $G_{xy}(f)$, $G_{xx}(f)$ and $G_{yy}(f)$ are the theoretical cross- and autospectral densities, respectively, at frequency f. The MSC can be estimated by ensemble averaging over various data segments, or by band averaging over adjoining frequency components by a suitable spectral window, of the sample spectra to yield estimate C^2 of γ^2. Both the MSC and its estimators are bounded by zero and unity. The necessity for determining smoothed estimators of (1) is described in detail in [1].

The MSC is a very useful indicator of various properties of the linear relationship between $x(t)$ and $y(t)$, that is, of the coherent common power between the two measured signals. A nonunity value infers either: 1) noise on $x(t)$ and/or $y(t)$, 2) the system relating $x(t)$ to $y(t)$ is nonlinear, or 3) that there are processes other than $x(t)$ and $y(t)$ involved.

However, it is relatively well known that the estimators of (1) are biased estimators. For example, for the case of smoothing by ensemble averaging, and assuming there to be no bias due to a misalignment [2], Nuttall and Carter [3] have shown that the bias of C^2 is given by

$$B(C^2) = E[C^2] - \gamma^2 \simeq \frac{1}{N}(1-\gamma^2)^2\left(1+\frac{2\gamma^2}{N}\right) \tag{2}$$

where γ^2 is the theoretical MSC, C^2 is the estimated MSC, and N is the number of time data segments employed. The estimator C^2 of MSC γ^2 does not possess a probability distribution function (PDF) that has a normal (Gaussian) form, thus confidence limits and other statistical descriptors cannot be easily calculated (see [4] for graphs of the confidence bounds of the MSC at the 80 percent and 95 percent levels).

II. Normalized Transformed Magnitude Coherence Function (NTMCF)

It is suggested in [5] that application of R. A. Fisher's Z-transformation [6] to the positive square root of the estimate of the MSC, called the magnitude coherence (MC), yields a function that has a nearly normal PDF. This *transformed MC* function TMC is given by

$$T(f) = \operatorname{arctanh}(|\gamma(f)|). \tag{3}$$

Its estimate $\hat{T} = \operatorname{arctanh}(C)$, has a variance of

$$\sigma^2_{\hat{T}} = \operatorname{var}(\hat{T}(f)) \simeq \frac{1}{n-2} \tag{4}$$

[7] where n is the number of degrees of freedom associated with the estimate. For ensemble averaging with nonoverlapping data sets, $n = 2N$. Empirical studies by [8] have confirmed that this transformation is valid for $n > 20$ with $0.3 < \gamma^2 < 0.98$. The validity of the transformation may be extended to a larger range of γ^2 and for $n > 8$ if the estimate of the MC is first corrected for bias. Recent related work is reported in [9].

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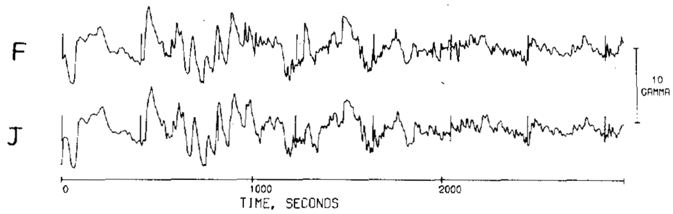

Fig. 1. Simultaneous recordings of the northward-directed component of the Earth's time-varying magnetic field as detected by a wire-suspended (J) and a fluxgate (F) magnetometer.

The TMC does not however lend itself to be used as a quantitative comparison between two estimates at the same f obtained with differing N's, i.e., between two estimates with differing n's. In order to facilitate such a comparison, it is suggested that the TMC estimate be normalized by the arctanh of the expectation value of the MC for two totally uncorrelated data sets. This latter value is given by

$$\operatorname{arctanh}\left((E[C_u^2])^{1/2}\right) = \operatorname{arctanh}\left(\left(\frac{2}{n}\right)^{1/2}\right) = \left(\frac{2}{n}\right)^{1/2} \text{ for } n \text{ large} \tag{5}$$

from (2) as $\gamma^2 = 0$ for uncorrelated random processes. Hence, the estimate of the *normalized transformed magnitude coherence* function (NTMC) is defined by

$$\hat{N}(f) = \frac{\operatorname{arctanh}(C(f))}{\operatorname{arctanh}((2/n)^{1/2})} \tag{6}$$

with a variance given by

$$\sigma_{\hat{N}}^2 = \operatorname{var}(\hat{N}(f)) = \frac{1}{(n-2)\operatorname{arctanh}^2((2/n)^{1/2})} = 0.5 \text{ for } n \text{ large.} \tag{7}$$

The following properties of the NTMC make it more preferable to the estimate of the MSC or the estimate of the MC.

1) Its mean is normalized to unity for two uncorrelated signals, hence indicating directly the coherent-to-incoherent common signal ratio.

2) Its PDF is more nearly normal.

3) Statistical parameters are easily calculated.

4) The variance is independent of the NTMC itself, and is a value of 0.5 for N large, i.e., independent of N also.

In order to demonstrate the use of the NTMC compared with the MSC, Fig. 1 illustrates two recordings of the northward-directed magnetic field variation with time as detected by two instruments of fundamentally very different operating principle (J, Jolivet torsion-band magnetometer; F, Fluxgate magnetometer) at the same locality and time. The estimate of the MSC spectrum was derived by smoothing over adjoining sample spectral estimates with the Bartlett spectral window [10] (the exact equivalence between ensemble averaging and smoothing the sample spectrum with the Bartlett window, of sinc-squared form, is detailed in [5]) of constant-Q, with Q = 0.3, to give the smoothed cross- and autospectral densities, then utilizing expression (1). The estimated MSC spectrum (C_{JF}^2) is illustrated in Fig. 2 where, for periods less than 30s, it would be concluded that the two series' do not well correlate due to the small values of C_{JF}^2. However, the estimates of the NTMC spectrum, as derived from C_{JF}^2 after transformation then normalization according to (6), are for periods in the range 15–20 s all greater than 5, with a very clear maximum of 8.7 at 65 s. The 95 percent confidence intervals of the estimates of N_{JF}, as derived from $1.96\,\sigma_N/(n)^{1/2}$, are plotted as

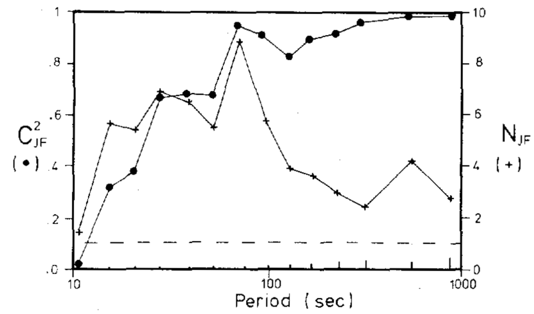

Fig. 2. Estimates of the MSC (C_{JF}^2) and the NTMC (N_{JF}) spectra between the two magnetic traces illustrated in Fig. 1. The 95 percent confidence intervals for the estimates of N_{JF} are given by the vertical bars at the base of the figure, and the expectation value ($E[N]$) for uncorrelated data (i.e. unity) is shown by the dashed line.

vertical bars in the lower part of the figure, from which it is conclusive that the coherence between the traces is statistically meaningful to such short periods as 15s.

III. Extension to Multivariate System Coherence Functions

The suggestion made in Section II can be generalized to define any estimate of a normalized transformed coherence function as being given by

$$\hat{N} = \frac{\operatorname{arctanh}(C)}{\operatorname{arctanh}((E[C_u^2])^{1/2})} \tag{8}$$

where C is the estimated magnitude coherence (ordinary, multiple, partial, etc.) and $E[C_u^2]$ is the expectation value of the estimator of the coherence function when all processes involved are totally uncorrelated. For example, for a p input-single output linear system, the estimator of the multiple coherence function has an expectation value for totally uncorrelated data of

$$E[C_u^2]_{\text{mult}} = \frac{p}{n-2p} \tag{9}$$

[8], and hence the normalized transformed multiple coherence function is estimated by

$$\hat{N}_{\text{mult}} = \frac{\operatorname{arctanh}(C_{\text{mult}})}{\operatorname{arctanh}((p/(n-2p))^{1/2})}. \tag{10}$$

Similar expressions are possible for the normalized transformed partial coherence functions.

A more complete exposition of these functions, and the use thereof, for a two input-single output linear system, is given in [11].

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References

[1] J. S. Bendat and A. G. Piersol, *Random Data: Analysis and Measurement Procedures.* New York: Wiley, 1971, pp. 193–196.

[2] G. C. Carter, "Bias in magnitude-squared coherence estimation due to misalignment," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-28, pp. 97–99, Feb. 1980.

[3] A. H. Nuttall and G. C. Carter, "Bias of the estimate of the magnitude-squared coherence," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-24, pp. 582–583, Dec. 1976.

[4] E. H. Scannell Jr. and G. C. Carter, "Confidence bounds for the

magnitude-squared coherence estimates," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-26, pp. 475-477, Oct. 1978.
[5] G. M. Jenkins and D. G. Watts, *Spectral Analysis and its Applications.* San Francisco: Holden-Day, 1968, pp. 239-243, 379.
[6] A. Hald, *Statistical Theory with Engineering Applications.* New York: Wiley, 1952, pp. 608-609.
[7] R. K. Otnes and L. Enochson, *Digital Time Series Analysis.* New York: Wiley-Interscience, 1972, pp. 352-353.
[8] L. D. Enochson and N. R. Goodman, "Gaussian approximation to the distribution of sample coherence," paper A.F.F.D.L.T.R. 65-57 Air Force Flight Dynamics Lab., Wright-Patterson A.F.B., OH, Feb. 1965.
[9] A. H. Nuttall and G. C. Carter, "An approximation to the cumulative distribution function of the magnitude-squared coherence function," *IEEE Trans. Acoust., Speech, Signal Processing*, to be published.
[10] M. S. Bartlett, *An Introduction to Stochastic Processes with Special Reference to Methods and Applications.* Cambridge: Cambridge Univ. Press, 1953.
[11] A. G. Jones, *Geomagnetic Induction Studies in Southern Scotland,* Ph.D. dissertation, Univ. Edinburgh, Edinburgh, Scotland, 1977.

Hyperstability and Adaptive Filtering

M. G. LARIMORE

Abstract—**Recently, results from the area of system identification concerning the concept of system hyperstability have been applied to the analysis of adaptive filtering configurations. In particular, this analysis tool has allowed the development of a family of convergent adaptive recursive digital filtering algorithms, where the use of conventional gradient analysis techniques had proven insufficient. This correspondence provides a somewhat informal explanation of hyperstability analysis as might be useful to the adaptive signal processing community, and clarifies its implications and limitations.**

I. Introduction

The concept of hyperstability has appeared primarily in the context of control system analysis, providing a powerful means of assuring stability in a broad class of systems. For example, it has recently appeared in the context of output error identification via the model reference adaptive structure [1]. It is interesting to note that the signal processing community has had little exposure to this analysis tool. Yet, the study and development of adaptive signal processing techniques has frequently been hampered by the complexity of time-varying parameter behavior. Convergence analysis of techniques commonly in use in signal processing relies largely on gradient approximation methods. Two innovative adaptive filtering techniques of current interest involve the lattice and feedback structures, which do not lend themselves to convergence analysis by conventional methods. This paper provides a somewhat informal explanation of hyperstability analysis as might be useful to the signal processing community, and hopefully will clarify its features and limitations.

II. System Hyperstability

For our purposes, we draw on published work concerning the continuous time case [2], providing the obvious modifications of definitions and properties when necessary. A rigorous treatment of the discrete time case can be found in [3]. Consider a discrete time-invariant linear system, controllable and observable, with a state realization,

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k). \quad (1)$$

For signal processing applications, its input/output relation is often an adequate characterization:

$$G(z) = \frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D. \quad (2)$$

Note we have restricted the discussion to the scalar case for convenience only; the multiinput, multioutput case follows with the usual alterations [3]. If we restrict the driving input $u(k)$ to a particular class of well-behaved sequences, where

$$\sum_{l=0}^{K_0} u(l)\, y(l) < \gamma_0^2, \qquad \forall K_0 > 0 \quad (3)$$

then the system is said to be *asymptotically hyperstable* if

$$\lim_{k \to \infty} \|x(k)\| = 0. \quad (4)$$

(Strictly speaking, *hyperstability* merely implies boundedness of the internal state; in this paper, we simply drop the modifier *asymptotically* for the sake of informality.) From an output point of view, $y(k)$ will also be bounded, and in the case of $D = 0$, i.e., $G(z)$ has a proper rational form, $y(k)$ must also converge to zero.

The hyperstability theorem of Popov [4] claims that the system $G(z)$ is hyperstable if and only if $G(z)$ is strictly positive real (SPR), i.e., on the unit circle contour,

$$\text{Re}\,[G(z)] > 0 \qquad z = e^{j\theta}. \quad (5)$$

Stated concisely, if a system has a SPR transfer function $G(z)$, then for *any* input $u(k)$ satisfying (3), the output $y(k)$ will remain bounded. This is a slightly stronger variation of bounded input/bounded output (BIBO) stability, since it allows certain divergent inputs.

Anderson notes [2] that this property is a specialization of the concept of system passivity. As a familiar physical example, consider an input port of a passive network, with a driving current and a voltage response. It is well known that a passive driving point impedance is positive real [5]. To be precise, this discussion deals with a *strictly* passive network where proper placement of resistive elements assures internal dissipation. Then, there exists a state realization with each state variable corresponding to an energy storage element. If the energy injected into the port's driving point

$$E = \int_0^T v(t)\, i(t)\, dt \quad (6)$$

is bounded, intuitively we would expect (as predicted by the hyperstability theorem) that the energy stored internally would eventually be dissipated, i.e., $\|x(t)\| \to 0$.

III. Closed-Loop Stability

In the form outlined in the previous section, the system property of hyperstability is of little practical use. Its power becomes evident when we consider an input sequence $u(k)$ derived as a general memoryless function of the output $y(k)$, possibly time varying and nonlinear. Fig. 1 indicates this feedback

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