Geophys. J. R. astr. Soc. (1985) **80**, 551–554

Research note

Discrete polar motion equations

Clark R. Wilson *Center for Space Research, Institute for Geophysics, and Department of Geological Sciences, The University of Texas at Austin, Texas 78713, USA*

Accepted 1984 August 15. Received 1984 August 15; in original form 1983 November 16

Summary. A digital filter equation is derived which is appropriate for predicting polar motion from excitation axis displacements, or for inferring the excitation axis changes from observed polar motion. The result differs from previously published equations in its phase response. Two additional equations are presented which are useful if samples of the excitation and polar motion functions are required to be at the same time values.

This discussion is concerned with the problem of designing equations which may be used in the study of the Earth's polar motion. The objectives are to predict polar motion from time samples of variations in the excitation axis (equivalent to the principal axis when there is no motion relative to the Earth); or to estimate variations in the excitation axis from samples of the pole position. Let the motion of the pole be represented by $M(t)$, a continuous complex function of time t, with real part denoting displacements along the Greenwich Meridian, and imaginary part displacements along 90° east longitude. Similarly, $X(t)$ is a complex function describing the motion of the excitation axis. Finally, F_c is Chandler's frequency of the free nutation (about 0.843 cycle yr^{-1}), $1/Q_c$ is the dissipation factor (Q_c is the quality factor) of the free nutation, and $\sigma_c = 2\pi F_c(1 + i/2Q_c)$ is a complex frequency which succinctly describes the Chandler frequency and damping.

A differential equation describing the relationship between $M(t)$ and $X(t)$ is derived from Euler's rigid body equations with corrections to account for dissipation and for the lengthening of the free period due to the non-rigid nature of the Earth. Letting $i = \sqrt{-1}$, the equation is:

$$X(t) = \frac{i}{\sigma_c}\,\frac{dM(t)}{dt} + M(t). \tag{1a}$$

The transfer function of this equation, which is the ratio of the Fourier transform of $M(t)$ to that of $X(t)$ at frequency f, is

$$\frac{\sigma_c}{\sigma_c - 2\pi f}\,. \tag{1b}$$

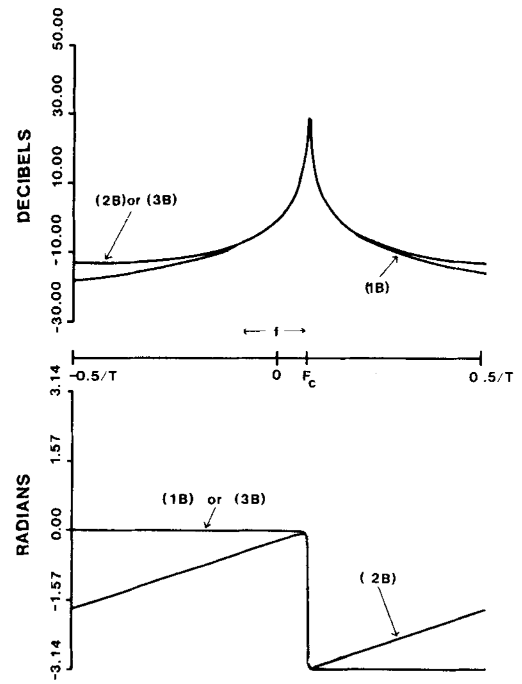

Figure 1. Transfer functions (1b) and (2b) displayed as amplitude in decibels ($20 \log_{10}$ of the modulus of the transfer function) and phase (arc tangent of the ratio of imaginary to real parts). Calculations were done for the case $T = 1$ month, $F_c = 0.843$ cycle yr^{-1}, and $Q_c = 100$.

The phase and amplitude are displaced in Fig. 1 as a function of frequency.

A discrete version of equation (1a) was presented by Jeffreys (1940), which Wilson & Haubrich (1976) expressed in complex notation as

$$X_t = \frac{i}{\sigma_c T} \quad M_t - \exp(i\sigma_c T) M_{t-T} \tag{2a}$$

where the subscript t denotes discrete samples of the continuous functions at time intervals T. The transfer function of equation (2a) is

$$\frac{-i\sigma_c T}{1 - \exp[i(\sigma_c - 2\pi f)T]} \tag{2b}$$

for which amplitude and phase are also shown in Fig. 1. Modern discussions of digital filter design would describe the transition from (1b) to (2b) as an example of the matched Z transformation (Tretter 1976, p. 212), which involves mapping the complex f plane pole in (1b) at $f = \sigma_c/2\pi$ into a pole in the complex Z plane $[Z = \exp(-2\pi i f T)]$ at the location $\exp(i\sigma_c T)$. This filter design method works quite well for the case of resonant behaviour (large Q_c) such as the Earth exhibits near the Chandler frequency.

It is evident from Fig. 1 that the amplitude of (2b) is a close match to (1b), but that the phase functions are not in good agreement, except near $f = F_c$. The phase discrepancy is apparently zero at F_c and linear in frequency, suggesting that a simple linear phase shift should correct the problem. Multiplication of (2b) by the quantity $\exp[i\pi(F_c - f)T]$ produces a discrete equation, and associated transfer function:

$$X_t = \frac{i\exp(-i\pi F_c T)}{\sigma_c T}\left[M_{(t+T/2)} - \exp(i\sigma_c T)\,M_{(t-T/2)}\right] \quad (3a)$$

$$\frac{-i\sigma_c T\exp[i\pi(F_c - f)T]}{1 - \exp[i(\sigma_c - 2\pi f)T]}. \quad (3b)$$

The amplitude of (3b) is the same as that of (2b) but the phase is now virtually indistinguishable from that of (1b) as shown in Fig. 1.

At the Chandler frequency the phase of (1b) and (3b) are precisely the same and the average phase difference at all other frequencies is very small (3×10^{-4} rad). At the Chandler frequency the amplitude of (3b) is slightly larger than that of (1b) by a factor

$$\frac{\pi F_c T/Q_c}{1 + 2\pi F_c T/Q_c - \exp(F_c T/Q_c)}$$

which approaches 1 as Q becomes infinite. However, even with Q_c as low as 10, this factor involves an error of 0.002 dB and is thus completely negligible. At other frequencies, the amplitude of (3b) appreciably differs from (1b) as shown in Fig. 2.

Equation (3a) permits recovery of the excitation axis displacements at times mid-way between samples of $M(t)$. However, if one is interested in recovering samples of $X(t)$ at the sampling times of $M(t)$, then estimates of $M_{(t+T/2)}$ and $M_{(t-T/2)}$ may be obtained by

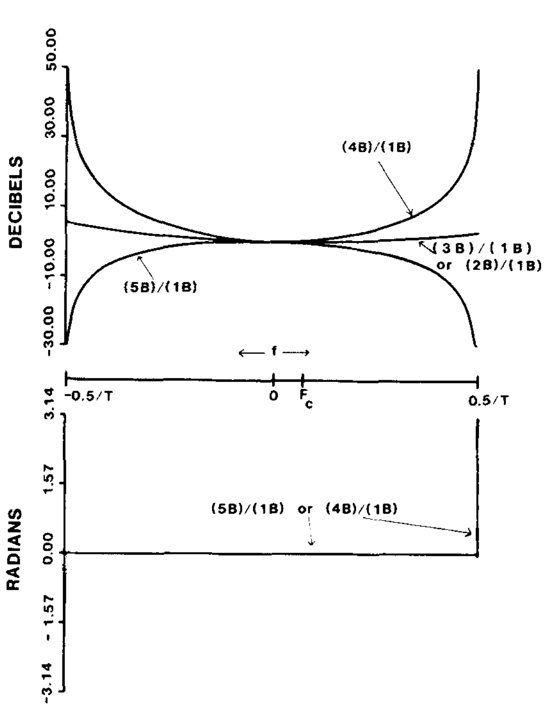

Figure 2. Transfer function ratio (4b)/(1b) and (5b)/(1b). The amplitude of the ratio is displayed in decibels. The phase of the ratio is the phase difference, which averages about 3×10^{-4} except near the Nyquist frequencies, $\pm 0.5/T$. The phase difference of (3b)/(1b) remains small even at the Nyquist frequencies.

linear interpolation of M_t, $M_{(t-T)}$, $M_{(t+T)}$ to yield

$$X_t = \frac{i\exp(-i\pi F_c T)}{2\sigma_c T} [\![M_{t+T} + [1 - \exp(i\sigma_c T)]\, M_t - \exp(i\sigma_c T)\, M_{t-T}]\!] \tag{4a}$$

$$\frac{-2i\sigma_c T\exp[i\pi(F_c - 2f)T]}{1 + [1 - \exp(i\sigma_c T)]\exp(-2\pi ifT) - \exp[i(\sigma_c - 4\pi f)T]} \tag{4b}$$

with the transfer function given by (4b). The amplitude and phase differences between (4b) and (1b) are plotted in Fig. 2. Equation (4a) would not be suitable for predicting M_t from X_t since it has undesirable amplification at the Nyquist frequencies. However, it would be useful for estimating X_t from M_t, since the transfer function would then be the reciprocal of (4b) and thus would attenuate Nyquist frequency variations, a desirable feature if the M_t values are corrupted by noise.

For the task of predicting M_t and X_t, one may employ a similar method to obtain:

$$M_t = \frac{-i\sigma_c T\exp(i\pi F_c T)}{2} [X_t + X_{t-T}] + \exp(i\sigma_c T) M_{t-T} \tag{5a}$$

$$\frac{-i\sigma_c T\exp(i\pi F_c T)\,[1 + \exp(-2\pi ifT)]}{2(1 - \exp[i(\sigma_c - 2\pi f)T])} \tag{5b}$$

with the transfer function given by (5b). The amplitude and phase difference between (5b) and (1b) are plotted in Fig. 2.

There are large amplitude differences between (4b) or (5b) and the exact amplitude (1b) as shown in Fig. 2. These arise because the averaging used to obtain these equations has introduced a pole in the complex plane for (4b), and a zero for (5b). The location of this pole or zero is exactly on the unit circle (at $Z = -1$). It would be possible to move the pole or zero just outside the unit circle, on the real axis. The result would be more moderate amplitude behaviour, but less perfect agreement with the phase of (1b), near the Nyquist frequencies.

Acknowledgment

The author appreciates the support of the Owen Coates Fund of the Geology Foundation of the University of Texas at Austin in the preparation of the manuscript for publication.

References

Jeffreys, H., 1940. The variation of latitude, *Mon. Not. R. astr. Soc.*, **100**, 139.

Tretter, S., 1976. *Introduction to Discrete Time Signal Processing*, Wiley, London.

Wilson, C. & Haubrich, R., 1976. Meteorological excitation of the Earth's wobble, *Geophys. J. R. astr. Soc.*, **46**, 707.