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Universality in sandpile models

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A classification of sandpile models into universality classes is presented. On the basis of extensive numerical simulations, in which we measure an extended set of exponents, the Manna two-state model [S. S. Manna, J. Phys. A. **24**, L363 (1991)] is found to belong to a universality class of random neighbor models which is distinct from the universality class of the original model of Bak, Tang, and Wiesenfeld [P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987)]. Directed models are found to belong to a universality class which includes the directed model introduced and solved by Dhar and Ramaswamy [D. Dhar and R. Ramaswamy, Phys. Rev. Lett. **63**, 1659 (1989)].

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The introduction of sandpile models as a paradigm of self-organized criticality by Bak, Tang, and Wiesenfeld (BTW) [1] stimulated numerous theoretical [2,3] and numerical studies [4–7]. In these models, which are defined on a lattice, grains are deposited randomly until the height at some site exceeds a threshold, and becomes unstable. "Sand" is then distributed to the nearest neighbors. As a result of this relaxation process neighboring sites may become unstable, resulting in a cascade of relaxations called *an avalanche*. It was observed that these models are self-driven into a critical state which is characterized by a set of exponents [1]. These include exponents that describe the distribution of quantities such as avalanche size and lifetime, and exponents which relate these properties of the dynamics. Large scale simulations of the BTW model [4] and some variants of it [8,9] were performed. The BTW model and the Manna two-state model were concluded to belong to the same universality class [8]. Christensen and Olami later introduced an extended set of exponents [7]. They measured the values of these exponents for the BTW model, and gave theoretical predictions and heuristic arguments for the values of some of the exponents. Continuous height models were also studied [10] and some aspects of universality were examined [11]. A sandpile model with a preferred direction was introduced and solved by Dhar and Ramaswamy [3].

In this paper we present simulation results which suggest a classification of sandpile models into universality classes. The Manna two-state model is found to belong to a universality class of random relaxation models which is distinct from the BTW universality class. We first describe the different models, and define the properties of avalanches, with the exponents characterizing them. The models are defined on a d dimensional lattice of linear size L. Each site is assigned a dynamic variable $E(\mathbf{i})$ which represents some physical quantity such as energy, stress, etc. In a critical height model a configuration $\{E(\mathbf{i})\}$ is called *stable* if for all sites $E(\mathbf{i})<E_c$, where E_c is a threshold value. The evolution between stable configurations is by the following rules.

(i) Adding energy. Given an arbitrary stable configuration $\{E(\mathbf{j})\}$ we select a site $\mathbf{i}$ at random and increase $E(\mathbf{i})$ by some amount δE. When an unstable configuration is reached rule (ii) is invoked.

(ii) The relaxation rule. If the dynamical variable at site $\mathbf{i}$ exceeds the threshold E_c, relaxation takes place, whereby energy is distributed in the following way:

$$\begin{aligned} E(\mathbf{i}) &\to E(\mathbf{i}) - \sum_{\mathbf{e}} \Delta E(\mathbf{e}), \\ E(\mathbf{i}+\mathbf{e}) &\to E(\mathbf{i}+\mathbf{e}) + \Delta E(\mathbf{e}), \end{aligned} \tag{1}$$

where $\mathbf{e}$ are a set of (unit) vectors from the site $\mathbf{i}$ to some neighbors. As a result of the relaxation the dynamic variable in one or more of the neighbors may exceed the threshold. The relaxation rule is then applied until a stable configuration is reached. The sequence of relaxations is an avalanche which propagates through the lattice.

The parameters δE and E_c are irrelevant to the scaling behavior [2,11]. Thus the only factor determining the exponents is the vector ΔE, to be termed *relaxation vector*. For a square lattice with relaxation to nearest neighbors it is of the form $\Delta E=(E_N,E_E,E_S,E_W)$, where E_N for example is the amount transferred to the northern nearest neighbor. The original BTW model is given by the vector (1,1,1,1). The relaxation in the directed model of Dhar and Ramaswamy [3] is specified by any vector with 1's in two adjacent directions and 0's in the two other directions, such as (0,0,1,1). In a random relaxation model a set of neighbors is randomly chosen for relaxation. Such a model is specified by a set of relaxation vectors, each vector being assigned a probability for its application. As an example, a possible realization of a two-state model makes use of the six relaxation vectors (1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), and (0,0,1,1), each one applied with a probability of 1/6. In Manna's two-state model [8] the variable is decreased to zero on relaxation, with sand distributed randomly among the nearest neighbors. We define a current

$$\mathbf{J}[\Delta E] = \sum_{\mathbf{e}} \Delta E(\mathbf{e})\mathbf{e}, \tag{2}$$

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