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Slip, Stress Drop and Ground Motion of Earthquakes: A View from the Perspective of Fractional Brownian Motion

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Abstract—The characteristics of slip and stress drop distributions accompanying earthquakes are explored from the perspective of fractional Brownian motion (fBm). Slip and stress drop distributions are assumed to be processes of fBm. The Hurst exponent (H), which reveals the roughness of a random process of fBm, is first estimated from ten inferred slip maps for six crustal earthquakes occurring in California. The relationships between the Hurst exponents with respect to static slip (H_u), stress (H_τ), static stress drop ($H_{\Delta\sigma}$) and slip velocity ($H_{\dot{u}}$) are then established following ANDREWS (1980). They are found to be $H_{\Delta\sigma} = H_\tau = H_u - 1 = H_{\dot{u}} - 0.5$. Empirically, H_u is recognized as being about 1 which, according to the theory of fBm, implies that the static slip distribution of an earthquake is just on the margin between being and not being self-similar, depending on the individual case. Cases where H_u is less than 1 (i.e., self-similar) suggest that $H_{\Delta\sigma} < 0$ (i.e., the distribution of static stress drop diverges), which is, in light of fBm, invalid. One possible explanation for this paradox is that H_u is less than 1 in crustal earthquake phenomena only over a certain specific bandwidth of wavenumbers, or it could be that the relation $H_{\Delta\sigma} = H_u - 1$ is not valid, which implies that static stress drop in the wavenumber domain is not the product of stiffness and slip as described in ANDREWS (1980). It could be that some different physics apply over this particular bandwidth. In such cases, multi-fractals may be a better way to explore the characteristics of the Hurst exponents of slip. In general, static stress drop and stress distributions are more likely to be self-similar than static slip distribution. $H_u \cong 1$ and $H_{\Delta\sigma} \cong 0$ are good first approximations for the slip and stress drop distributions. The spectrum of ground motion displacement falls off as $\omega^{-(H_{\Delta\sigma}+2)}$ with $H_{\Delta\sigma} \cong 0$, consistent with an ω^{-2} model of the earthquake source.

Key words: Slip distribution, stress drop, strong ground motion, Hurst exponent, fractional Brownian motion, fractal dimension.

Introduction

Many patterns of nature are so irregular and fragmented that standard Euclidean geometry cannot delineate them. Nature exhibits not simply a higher degree but also an altogether different level of complexity. Those forms that Euclid sets aside as being ‘‘formless’’ were identified to be a family of shapes that MANDELBROT (1982) called *fractals*. An ordinary cauliflower represents an excellent example of a formless shape that can be suitably described by fractals: ‘‘The cauliflower head contains branches or parts, which when removed and compared with the

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whole are very much the same, only smaller'' (PEITGEN *et al.*, 1992). The character that the shapes of cauliflower, or the like, tend to be scaling, is referred to as *self-similarity*, in a rough sense. This implies that the degree of irregularity and fragmentation is identical on all scales. The trace of one-dimensional Brownian motion is one of the self-similar fractals, which is one special case of a generalized random function introduced by Mandelbrot who named it *fractional Brownian motion* (MANDELBROT and VAN NESS, 1968; MANDELBROT, 1982). Earthquakes are one such type of highly irregular phenomena in nature. For several decades now, the concept of fractals has been applied to earthquake phenomena, first implicitly and later explicitly, by researchers in the seismological community.

''Similarity'' was first introduced by TSUBOI (1956) to relate earthquakes of different sizes by a one-parameter model. Ever since, the assumption of self-similarity has been widely applied to seismic sources (e.g., AKI, 1967, 1972; HANKS, 1979; ANDREWS, 1980, 1981; FRANKEL, 1991; HERRERO and BERNARD, 1994; ZENG *et al.*, 1994). Inspired by MANDELBROT (1977), ANDREWS (1980, 1981) applied the concept of ''self-similar irregularity'' to earthquake phenomena and suggested that ''fractals can also describe the occurrence and mechanics of earthquakes''. FRANKEL's (1991) source model assumes a self-similar distribution of subevents and suggests that the ω^{-2} spectral falloff can be attributed to scale-invariant strength along fault zones, corresponding to constant stress drop scaling. Following Frankel's self-similar model, ZENG *et al.* (1994) proposed a composite source model for the synthesis of realistic strong ground motions. HERRERO and BERNARD (1994), assuming the spectral amplitude of slip is independent of the size of the seismic source, proposed a k-square model for the slip spectrum of the ruptured fault, which also results in the ω^{-2} model for the radiated body waves.

It seems that fractals play fruitful roles in illustrating earthquake phenomena. However, the adequacy of doing that has never been tested before. Whether or not there exists a universal scaling (or self-similar) ''rule'' governing earthquake phenomena is of interest to the seismological and earthquake engineering communities. Recently, inferred slip maps have become available for some events, namely, the 1979 Imperial Valley earthquake (HARTZELL and HEATON, 1983), the 1984 Morgan Hill earthquake (HARTZELL and HEATON, 1986; BEROZA and SPUDICH, 1988), the 1987 Whittier Narrows earthquake (ZENG *et al.*, 1993), the 1989 Loma Prieta earthquake (BEROZA, 1991; WALD *et al.*, 1991), the 1992 Landers earthquake (WALD and HEATON, 1994a; COHEE and BEROZA, 1994), and the 1994 Northridge earthquake (WALD and HEATON, 1994b; WALD *et al.*, 1996; ZENG and ANDERSON, 1996). It is felt that it is of great benefit to reveal the slip characteristics of these events and to justify the application of the concept of fractals to earthquake phenomena.

In this paper, the author takes a fractional Brownian motion (fBm) perspective to explore the characteristcs of the slip and stress drop distributions of an earthquake. From the viewpoint of fBm, a parameter describing the ''roughness'' of the slip distribution is estimated from the inferred slip maps, in which asperities are clearly observed. It has been demonstrated that this parameter, referred to here as the ''Hurst exponent'' (HURST *et al.*, 1965; MANDELBROT, 1977), plays a crucial role in earthquake phenomena. The Hurst exponents with respect to slip (H_u) are first estimated from ten slip models. The results reveal that H_u is in the neighborhood of 1, which means that the slip distribution in the space domain may be either fractal ($H_u < 1$) or differentiable ($H_u \geq 1$), depending on the individual events. Assuming that the stress in the seismogenic zone of the crust is a random process described by fBm, the stress drop can similarly be described by fBm, which implies that H_τ (where τ represents stress) is equal to $H_{\Delta\sigma}$ (where $\Delta\sigma$ represents stress drop). Based on ANDREWS (1980), the relation $H_{\Delta\sigma} = H_u - 1$ is reached. This suggests that $H_{\Delta\sigma}$ is, in general, close to 0. However, from the theory of fBm, it cannot be equal to or less than 0. An interesting issue thus raised in this study is that in these cases where there is a possibility that $H_u < 1$, the unassailableness of the relation $H_{\Delta\sigma} = H_u - 1$ is put into question. In light of this, it is suggested that by estimating H_u in different bandwidths of wavenumber, multi-fractals may be an alternative solution to this issue. As a rule, $H_u \cong 1$ and $H_{\Delta\sigma} \cong 0$ are first good approximations for slip and stress drop distributions, which indicates that a falloff of ω^{-2} is a good approximation for ground motion simulations.

Fractional Brownian Motion

In one dimension, a random process X of a real variable t (time or distance) is called *fractional Brownian motion* (MANDELBROT and VAN NESS, 1968; MANDELBROT, 1982) if the increment $X(t_2) - X(t_1)$ has a Gaussian distribution with mean zero and variance:

$$\mathrm{Var}(X(t_2) - X(t_1)) = |t_2 - t_1|^{2H} \sigma^2_{x_1 - x_0}, \tag{1}$$

where H is referred to as the ''Hurst exponent'' with values between 0 and 1 (FEDER, 1988, pp. 170–183; SAUPE, 1988; VOSS, 1988; PEITGEN *et al.*, 1992, p. 415); and $\sigma^2_{x_1 - x_0}$ is the variance of the difference $X(1) - X(0)$ when $t_2 - t_1$ equals 1 unit of the variable t.

The increments of X are *statistically self-affine* with H. That is,

$$X(t_0 + t) - X(t_0),$$

and

$$\frac{1}{s^H}(X(t_0 + st) - X(t_0)) \tag{2}$$

are statistically indistinguishable for any t_0 and $s > 0$, scaling differently in the t and X coordinates. The scaling property of fBm represented above is different from *statistical self-similarity*, which repeats its ''shape'' with the same magnifications in both the t and X directions. For the exponent $H = 0$, the variance is independent of scaling. X still ''looks'' the same for all $s > 0$. This means X can be expanded or contracted in the t coordinate by any factor, thus implying that a sample of X with $H = 0$ must densely fill up the plane on which X is shown. If $H = 1$, the opposite case prevails. For other values of the exponent, X behaves differently depending on H, such that:

- $H = \frac{1}{2}$; X is the ordinary Brownian motion which has independent increments.
- $H > \frac{1}{2}$; the increments of X have a positive correlation, i.e., if X increases for t_0, it tends to continue to increase for $t_0 + t$ $(t > 0)$.
- $H < \frac{1}{2}$; the opposite of the above holds, and X appears more erratic.

The Hurst exponent thus indicates the ''roughness'' of the curve depicted by X plotted against t. It is also related to the *fractal dimension*, D_f (MANDELBROT, 1977), by the equation:

$$D_f = n + 1 - H, \tag{3}$$

where n denotes the number of dimensions. In the one-dimensional case, $D_f = 2 - H$. The one-dimensional fBm of X is ''fractal'' when its fractal dimension is in the range of $1 < D_f < 2$.

For a sample of fBm in the multi-dimensional domain, its power spectrum is related to the Hurst exponent as follows (SAUPE, 1988):

$$S(k_1, \ldots, k_n) \propto \frac{1}{(\sqrt{k_1^2 + \cdots + k_n^2})^{2H+n}}. \tag{4}$$

In the one-dimensional case, the power spectrum falls off as $k^{-(2H+1)}$. Since the spectral amplitude is proportional to the square root of the power spectrum, the one-dimensional spectral amplitude of fBm falls off as $k^{-(H+0.5)}$, or as $k^{-(2.5-D_f)}$, which is consistent with that mentioned in SCHOLZ and AVILES (1986), if it is in terms of the fractal dimension.

The Hurst Exponent with Respect to Slip

Nature reveals fractal characteristics for many phenomena on earth (e.g., MANDELBROT, 1977, 1982; PEITGEN and RICHTER, 1986). If the fractal character is embedded in the fault slip distribution, the *roughness* of the slip curve can be represented by the Hurst exponent H, and the curve itself can be simulated by fBm. This raises the question as to the degree of roughness of the fault slip, or in other words, the value of H. The answer may lie in the inferred slip maps of earthquake events.

Let $u(x, y)$ denote the two-dimensional slip distribution of the fault. The slip profile $u(x)$ (at some ''depth'' y) delineated by one-dimensional fBm can be transformed from the space domain into the wavenumber domain. For the wavenumber k greater than the corner frequency k_{0x} (which is equal to the inverse of the rupture length along the x direction), according to Equation (4), its power spectrum falls off as $k^{-(2H+1)}$ or by Equation (3) with $n = 1$, as $k^{-(5-2D_f)}$. If $H = 1$ (or equivalently, $D_f = 1$), the power spectrum falls off as k^{-3}, with $u(x)$ being nonfractal and differentiable. In contrast, if the power spectrum falls off flatter than k^{-3} with $0 < H < 1$, the slip profile $u(x)$ is not differentiable, but it is fractal with $1 < D_f < 2$.

To investigate the characteristics of the Hurst exponent H for the inferred slip maps, the assumption is made here that the falloff of the spectral amplitude of slip in two dimension ($n = 2$) is as follows:

$$|U(k_x, k_y)| = \frac{AMP}{1 + \left(\sqrt{\dfrac{k_x^2 + k_y^2}{k_{0x}k_{0y}}}\right)^{H+1}}, \tag{5}$$

where $U(k_x, k_y)$ is the two-dimensional Fourier transform of slip $u(x, y)$; k_{0x} and k_{0y} are the corner frequencies along the x and y coordinates, respectively; and AMP is equal to $|U(0, 0)|$. For wavenumbers $k_x \gg k_{0x}$ and $k_y \gg k_{0y}$:

$$|U(k_x, k_y)| \cong \frac{AMP}{\left(\sqrt{\dfrac{k_x^2 + k_y^2}{k_{0x}k_{0y}}}\right)^{H+1}}. \tag{6}$$

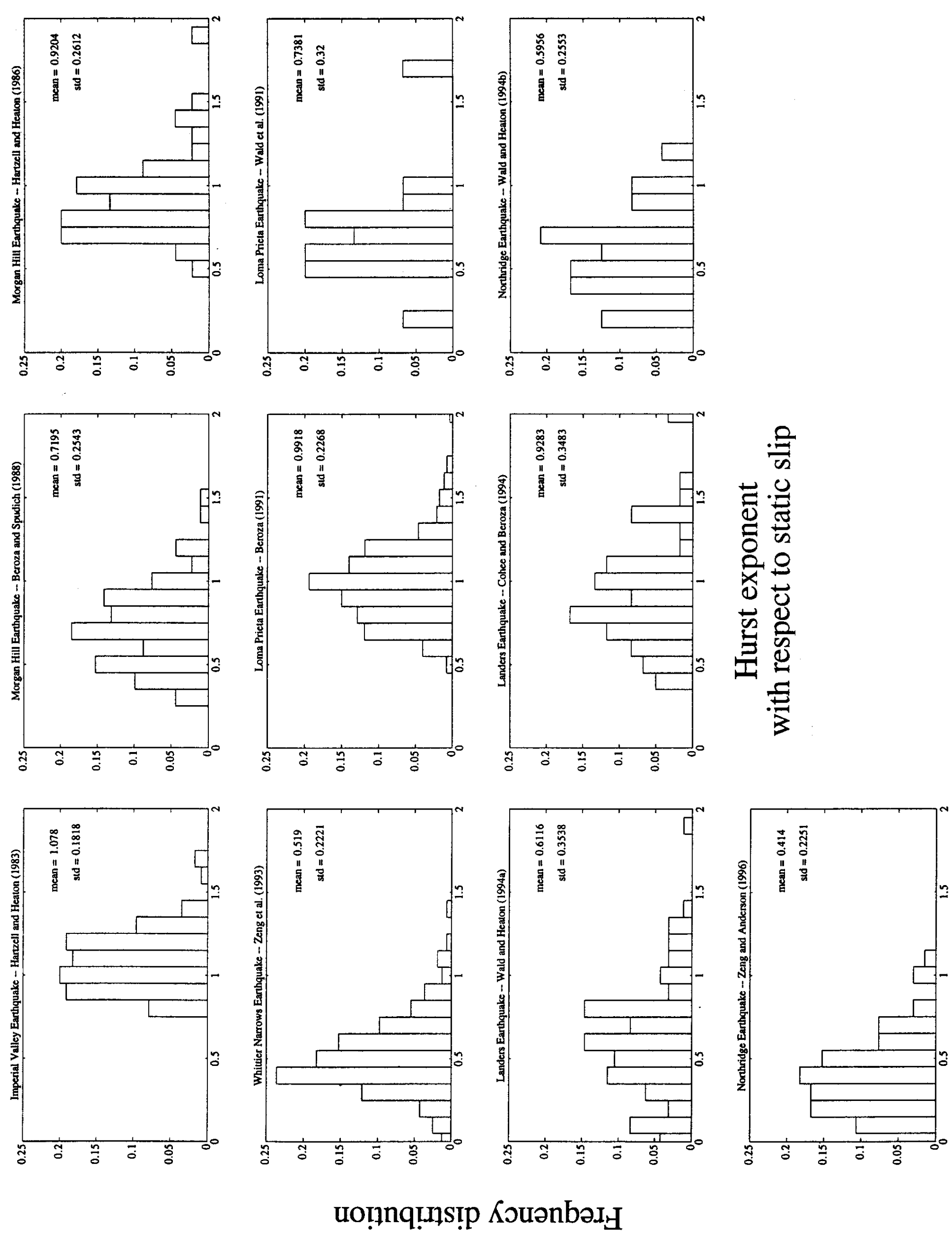
The rearrangement of Equation (6) yields the Hurst exponent:

$$H(k_x, k_y) = \frac{\log(AMP) - \log|U(k_x, k_y)|}{\log\left(\sqrt{\dfrac{k_x^2 + k_y^2}{k_{0x}k_{0y}}}\right)} - 1, \tag{7}$$

which, except for both k_x and $k_y = 0$, has a value corresponding to each pair of wavenumbers (k_x, k_y). These values of $H(k_x, k_y)$ are calculated from the ten slip models mentioned previously and listed in Table 1. Figure 1 shows the histograms and frequency distributions of the Hurst exponents $H(k_x, k_y)$ for wavenumbers $k_x \gg k_{0x}$ and $k_y \gg k_{0y}$. The mean values and standard deviations of H are also shown with each histogram.

TSAI (1992) evaluated the Hurst exponents applying Equation (7) and slip maps inferred from three events: the Imperial Valley, the Morgan Hill and the Loma Prieta earthquakes. Although the values of the Hurst exponents for these three earthquakes were generally close to 1.0, in this study, lower values of $0.40 < H < 0.67$ (see the results from Equation (7) in Table 1) are obtained with the availability of more slip maps from the Whittier Narrows, the Landers (except for the model by COHEE and BEROZA, 1994) and the Northridge earthquakes.

To avoid multiple estimates of H resulting from Equation (7), an alternative computation is implemented in this study in order to obtain a unique value of H for each slip model. Based on the linear regression of $\log|U(k_x, k_y)|$ on


Imperial Valley Earthquake -- Hartzell and Heaton (1983)
mean = 1.078
std = 0.1818
Morgan Hill Earthquake -- Beroza and Spudich (1988)
mean = 0.7195
std = 0.2543
Morgan Hill Earthquake -- Hartzell and Heaton (1986)
mean = 0.9204
std = 0.2612
Whittier Narrows Earthquake -- Zeng et al. (1993)
mean = 0.519
std = 0.2221
Loma Prieta Earthquake -- Beroza (1991)
mean = 0.9918
std = 0.2268
Loma Prieta Earthquake -- Wald et al. (1991)
mean = 0.7381
std = 0.32
Landers Earthquake -- Wald and Heaton (1994a)
mean = 0.6116
std = 0.3538
Landers Earthquake -- Cohee and Beroza (1994)
mean = 0.9283
std = 0.3483
Northridge Earthquake -- Wald and Heaton (1994b)
mean = 0.5956
std = 0.2553
Northridge Earthquake -- Zeng and Anderson (1996)
mean = 0.414
std = 0.2251
Hurst exponent
with respect to static slip
Frequency distribution

$\log(\sqrt{k_x^2 + k_y^2}/k_{0x}k_{0y})$ from Equation (6), the least-squares value of H can be obtained as follows:

$$H = \frac{\sum_{i:k_i > k_{0x}}^{N} \sum_{j:k_j > k_{0y}}^{M} A_{ij} \log\left|U(k_i, k_j)\right| - (\log AMP) \sum_{i:k_i > k_{0x}}^{N} \sum_{j:k_j > k_{0y}}^{M} A_{ij}}{\sum_{i:k_i > k_{0x}}^{N} \sum_{j:k_j > k_{0y}}^{M} A_{ij}^2} - 1,$$

$$A_{ij} = -\frac{1}{2}\log\left(\frac{k_i^2 + k_j^2}{k_{0x}k_{0y}}\right), \tag{8}$$

where N and M are integers such that k_N and k_M are the Nyquist wavenumbers along the x and y coordinates of the fault, respectively.

In addition to the mean values of H estimated from Equation (7), Table 1 also lists the fitted values of H estimated from Equation (8). The Hurst exponents of these two categories for each slip model are very close. Figure 2 exhibits the scattergram for the two categories of the Hurst exponents, together with the correlation coefficient which is equal to 0.995. The high correlation and linear relationship between these two types of estimates of H may result because the two methods are closely related mathematically, or because the Hurst exponents obtained by either method are very stable. To be further assured of the characteristics of the coseismic slip distribution in terms of H, the author performs additional computations below.

It should be noted that both Equations (7) and (8) represent methods based on the assumption that AMP, k_{0x} *and* k_{0y} are known quantities. In a different way, Equation (6) may be treated as a linear relationship of the form:

$$y = a + bx \tag{9}$$

between the variables $y = \log\left|U(k_x, k_y)\right|$ and $x = \log(\sqrt{(k_x^2 + k_y^2)/k_{0x}k_{0y}})$ with AMP, k_{0x} and k_{0y} being unknown. The value b can be determined from standard regression methods, and $H = -b - 1$. The standard error of the b values from the regression can also be determined, and hence, the standard deviation of H can be obtained. Given appropriate ranges of values for k_{0x} and k_{0y}, the optimal value of b is the one that corresponds to k_{0x} and k_{0y} which give the lowest standard error.

The least-squares values together with the statistical uncertainty in the estimates of H from Equation (9) are shown in Table 1 and are compared with those of the previous two methods. To provide some idea as to the quality of the slip data that are used in this study, the grid points and element sizes are also shown in the Table

Figure 1
Histograms and frequency distributions of the Hurst exponents $H(k_x, k_y)$ estimated from Equation (7) for the ten slip models with wavenumbers $k_x \gg k_{0x}$ and $k_y \gg k_{0y}$. Mean values and standard deviations of H are also shown in each histogram.

Table 1

Hurst exponents with respect to slip calculated from Equations (7), (8) and (9). Hurst exponents calculated from Equation (8) from the Landers earthquake using WALD *and* HEATON*'s (1994a) slip model are first averaged prior to the estimation of the gross mean and standard deviation. Those values in parentheses are not used in estimating the gross mean and standard deviation using Equation (9)*

Event	Grid points	Element size km × km	Eq. (7)	Eq. (8)	Eq. (9)
Imperial Valley (HARTZELL and HEATON)	44 × 11	1.0 × 1.0	1.078 ± 0.182	1.075	1.001 ± 0.189
Morgan Hill (BEROZA and SPUDICH)	61 × 11	0.5 × 1.0	0.720 ± 0.254	0.738	1.431 ± 0.135
Morgan Hill (HARTZELL and HEATON)	28 × 13	1.0 × 1.0	0.920 ± 0.261	0.922	1.578 ± 0.198
Whittier Narrows (ZENG *et al.*)	25 × 25	0.5 × 0.5	0.519 ± 0.222	0.533	0.901 ± 0.120
Loma Prieta (BEROZA)	81 × 15	0.5 × 10	0.992 ± 0.227	1.001	1.542 ± 0.086
Loma Prieta	12 × 8	3.3 × 2.5	0.738 ± 0.320	0.731	0.603 ± 0.426
(WALD *et al.*)	10 × 6			0.507	(0.420 ± 0.637)
Landers	9 × 6	3.0 × 2.5	0.612 ± 0.354	0.590	(0.067 ± 0.599)
(WALD and HEATON)	12 × 6			0.570	0.991 ± 0.570
Landers (COHEE and BEROZA)	28 × 6	3.0 × 3.0	0.928 ± 0.348	0.915	0.624 ± 0.531
Northridge (WALD and HEATON)	9 × 10	2.0 × 2.0	0.596 ± 0.255	0.585	(0.017 ± 0.412)
Northridge (ZENG and ANDERSON)	23 × 23	1.0 × 1.0	0.414 ± 0.225	0.416	0.876 ± 0.135
Mean			0.752	0.747	1.061
Standard deviation			0.220	0.224	0.372

for each slip model. Some estimated values of H (in parentheses) obtained from Equation (9) do not seem reasonable, and this author believes they are not reliable. It should be kept in mind that appropriate ranges of the values for k_{0x} and k_{0y} are assigned to each slip model. Unexpectedly, the estimates of H from Equation (9) for slip models that have low quality of resolution are mostly negative except for those which correspond to the corner frequencies that give the optimal values of H. This implies that with Equation (9), a limited resolution of the slip model with small grid points and of large element size may produce unreliable estimates of H. On the other hand, however, bad (negative) estimates of H are rarely seen from Equation (7) and never from Equation (8).

The variability of H among the ten slip models may have been caused by differences in the grid points and element sizes among slip maps, and in the seismic data and the inversion techniques that have been used by different groups of researchers. No matter which slip model has a better representation of the coseismic

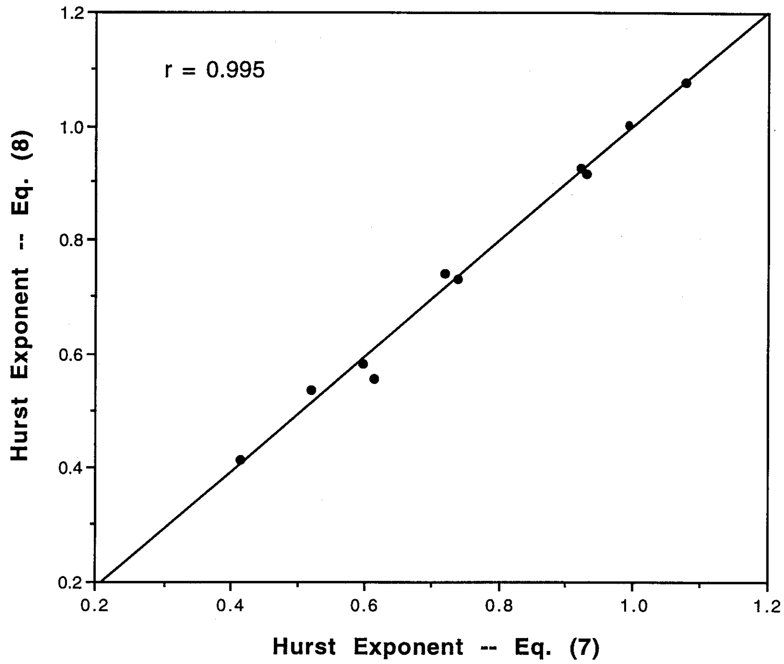


Figure 2
Scattergram of two category estimates of the Hurst exponents obtained from Equations (7) and (8), together with the correlation coefficient, *r*, which is equal to 0.995. The regression line is obtained by a least-squares fit to the data.

slip distribution, the mean value of H averaged over various slip models may provide a rough idea as to the coseismic slip of earthquake events. Both the gross means and standard deviations of H (Table 1) estimated from Equations (7) and (8) are similar with both equations and are approximately equal to 0.75 and 0.22, respectively. In contrast, if the unreliable estimates obtained from Equation (9) are ignored, the gross mean and standard deviations of H are 1.06 and 0.37, respectively.

It is worth noting that the optimal values of H obtained from Equation (9) mostly correspond to the corner frequencies which are much lower than those which are used with Equations (7) and (8). Another significant difference between the use of Equation (9) and that of Equations (7) and (8) is that the former treats AMP as an unknown variable, while the latter two treat it as a known quantity. Standard regression analysis using Equation (9) provides more information about the statistical uncertainty in the estimates of H. The standard deviation of the estimated H for each slip model varies from 0.086 to 0.570 (if unreliable estimates are excluded), depending on the quality of the slip data.

Though it may be asked which estimate of H better represents the characteristics of all crustal earthquakes, the response could be that it is too early to answer on the basis of the ten slip models. Although a gross mean of 0.75 from Equations (7) and

(8) is close to that of 1.06 from Equation (9), the difference in the implications of the Hurst exponent of slip between $H < 1$ and $H > 1$ is dramatic from the viewpoint of fBm. (Recall that if $H \geq 1$, the slip in the space domain is differentiable, whereas if $0 < H < 1$, the slip is fractal.) If Equation (9) is a better method for the estimation of H, it can be stated with little risk that the value of H is around 1 and varies with an uncertainty of ± 0.37 from event to event for different earthquakes, depending both on the characteristic of the crust and on the type of earthquake mechanism. To more clearly ascertain the *statistical* nature of the slip distribution, inferred slip maps with the highest possible quality resolution from more events are required.

The Hurst Exponents with Respect to Stress and Stress Drop

The relationship between the stress function, $\tau(x)$, and the stress drop function, $\Delta\sigma(x)$, can now be considered in terms of fBm in one dimension. Let τ^0 and τ^1 denote the initial stress before the earthquake and the final stress afterwards, respectively. The increment of stress drop at the two locations of x_1 and x_2 can be expressed as:

$$\begin{aligned}\Delta\sigma(x_2) - \Delta\sigma(x_1) &= [\tau^0(x_2) - \tau^1(x_2)] - [\tau^0(x_1) - \tau^1(x_1)] \\ &= [\tau^0(x_2) - \tau^0(x_1)] - [\tau^1(x_2) - \tau^1(x_1)]. \end{aligned} \tag{10}$$

If the stress $\tau(x)$ is fBm with the Hurst exponent H_τ, the increment of $\tau(x)$ is Gaussian with mean zero and its variance is:

$$\mathrm{Var}(\tau(x_2) - \tau(x_1)) = |x_2 - x_1|^{2H_\tau}\sigma^2_{\tau_1 - \tau_0}. \tag{11}$$

Following the above assumption, the stress field should be fBm no matter if it is before or after the earthquake. This being the case, it is logical to deduce that the increment of stress drop, $\Delta\sigma(x_2) - \Delta\sigma(x_1)$, is also a random process which can be expressed in terms of fBm and is Gaussian with mean zero and with variance:

$$\mathrm{Var}(\Delta\sigma(x_2) - \Delta\sigma(x_1)) = |x_2 - x_1|^{2H_{\Delta\sigma}}\sigma^2_{\Delta\sigma_1 - \Delta\sigma_0}, \tag{12}$$

where $H_{\Delta\sigma}$ is the Hurst exponent with respect to stress drop.

On the other hand, from the basic theory of probability (SPIEGEL, 1975), the variance in Equation (12) can also be described as follows:

$$\begin{aligned}\mathrm{Var}(\Delta\sigma(x_2) - \Delta\sigma(x_1)) &= \mathrm{Var}(\tau^0(x_2) - \tau^0(x_1)) + \mathrm{Var}(\tau^1(x_2) - \tau^1(x_1)) \\ &\quad - 2\,\mathrm{Cov}(\tau^0(x_2) - \tau^0(x_1), \tau^1(x_2) - \tau^2(x_1)). \end{aligned} \tag{13}$$

Following ANDREWS' (1981) argument that fluctuations over one thousand bars of stress may occur on the shorter wavelength scale, the last *covariance* term in Equation (13) can be ignored because of the low correlation between the increment of stress before and after the earthquake due to the high fluctuations of stress involved. (A covariance equal to zero is reasonable for the wavelength of interest here.)

Substituting Equations (11) and (12) into (13) yields:

$$\left|x_2 - x_1\right|^{2H_{\Delta\sigma}}\sigma^2_{\Delta\sigma_1 - \Delta\sigma_0} = \left|x_2 - x_1\right|^{2H_{\tau^0}}\sigma^2_{\tau^0_1 - \tau^0_0} + \left|x_2 - x_1\right|^{2H_{\tau^1}}\sigma^2_{\tau^1_1 - \tau^1_0}, \tag{14}$$

where H_{τ^0} and H_{τ^1} denote the Hurst exponents with respect to stress before and after the earthquake, respectively. By analogy with Equation (13), it is reasonable to determine that $\sigma^2_{\Delta\sigma_1 - \Delta\sigma_0} = \sigma^2_{\tau^0_1 - \tau^0_0} + \sigma^2_{\tau^1_1 - \tau^1_0}$. Thus, the expression on the left-hand side of Equation (14) becomes:

$$\left|x_2 - x_1\right|^{2H_{\Delta\sigma}}\sigma^2_{\Delta\sigma_1 - \Delta\sigma_0} = \left|x_2 - x_1\right|^{2H_{\Delta\sigma}}\sigma^2_{\tau^0_1 - \tau^0_0} + \left|x_2 - x_1\right|^{2H_{\Delta\sigma}}\sigma^2_{\tau^1_1 - \tau^1_0}. \tag{15}$$

It is apparent from the comparison of Equations (14) and (15) that $H_{\Delta\sigma} = H_{\tau^0} = H_{\tau^1}$. Consequently, the Hurst exponent with respect to stress drop is equivalent to that with respect to stress for both stress fields before and after the earthquake. That is:

$$H_{\Delta\sigma} = H_{\tau}. \tag{16}$$

This supports ANDREWS' (1980) assumption that the change in static shear traction in a single earthquake has the same spectrum as that of the total static shear traction on a fault that is strictly self-similar even though he later argued (ANDREWS, 1981) that this need not be true.

Equation (16) has yet a futher implication. Although the stress drop process is the difference in stress before and after the earthquake (i.e., $\Delta\sigma = \tau^0 - \tau^1$), if the stress function is fBm, the stress drop function is also fBm with the same Hurst exponent. Furthermore, both the stress and stress drop function have the same spectral amplitude which falls off as $k^{-(H_{\Delta\sigma} + 0.5)}$ in one dimension.

Implications for Slip, Slip Velocity, Stress Drop and Ground Motion

There is little doubt that the Hurst exponent plays an important role in revealing the characteristics of the slip and stress drop distributions on the fault surface. Not only does it exhibit the degree of roughness of a random process in the space domain, but it also dictates the steepness of spectrum falloff in the wavenumber domain. As discussed in the previous section, based on the assumption that stress is a random process described by fBm, it can be concluded that stress drop can also be described by fBm. Obviously, it is of great benefit therefore to explore the characteristics of the Hurst exponent, $H_{\Delta\sigma}$.

Let H_u denote the Hurst exponent with respect to slip. According to ANDREWS (1980, 1981), the static stress drop in the wavenumber domain is a multiplication of slip and stiffness. In his formulation, stiffness is proportional to the wavenumber k. It should be recalled that the one-dimensional spectral amplitude of the static slip falls off as $k^{-(H_u+0.5)}$ for wavenumbers higher than the corner frequency. As a result, the one-dimensional spectral amplitude of the static stress drop falls off as $k^{-(H_u-0.5)}$ in terms of the Hurst exponent of slip. On the other hand, however, in conjunction with the nature of fBm, the spectrum amplitude of the stress drop should fall as $k^{-(H_{\Delta\sigma}+0.5)}$ if the Hurst exponent is described in terms of the stress drop. Consequently, it is easy to understand that:

$$H_{\Delta\sigma} = H_u - 1. \tag{17}$$

ANDREWS (1981) showed that H_u is related to the Hurst exponent with respect to the slip velocity, $H_{\dot{u}}$, as follows:

$$H_{\dot{u}} = H_u - \frac{1}{2}, \tag{18}$$

and that the ground displacement transform at a point near the fault is proportional to:

$$|D(\omega)| \propto \omega^{-(H_{\dot{u}}+1.5)}. \tag{19}$$

For $H_{\dot{u}} = 1/2$ (i.e, $H_u = 1$), ground motion has an ω-square spectrum, which means that the ground acceleration is white noise with a flat spectral amplitude within certain bands between the corner and cutoff frequencies. ANDREWS (1981) favors $H_{\dot{u}} \cong 1/2$ and he has H_τ (following his notation) represent the Hurst exponent with respect to stress change, which is related to $H_{\dot{u}}$ as follows:

$$H_\tau = H_{\dot{u}} - \frac{1}{2}. \tag{20}$$

ANDREWS' (1981) H_τ is essentially equivalent to $H_{\Delta\sigma}$ in this article owing to the fact that

$$\begin{aligned} H_{\Delta\sigma} &= H_u - 1 \\ &= H_{\dot{u}} - \frac{1}{2} \\ &= H_\tau. \end{aligned} \tag{21}$$

The empirical observation that $H_u \cong 1$ (as reported by Equation (9) in Table 1) and that $H_{\Delta\sigma} \approx 0$ is implied, is consistent with Andrews' argument that $H_\tau \cong 0$ which implies $H_{\dot{u}} \approx 1/2$.

The choice of the Hurst exponent with respect to slip, slip velocity, stress or stress drop is rather arbitrary. FRANKEL (1991) selected stress on the fault as the self-similar random function (in other words, choosing the Hurst exponent with respect to stress, H_τ, with τ strictly representing stress, not stress drop), and accordingly he came to the conclusion that the ground displacement transform is proportional to:

$$|D(\omega)| \propto \omega^{-(1.5H_\tau+2)}. \tag{22}$$

For $H_\tau \cong 0$, he also had an ω-square model for ground motion. However, except for $H_\tau = 0$ and $H_{\dot{u}} = 1/2$ (or $H_{\Delta\sigma} = 0$), both of which imply ω-square models, the spectrum falloffs for both ANDREWS' (1981) and FRANKEL's (1991) ground motion spectra are not the same for Equations (19) and (22). For instance, if $H_{\dot{u}} = 1$ (i.e., $H_{\Delta\sigma} = 1/2$), Equation (19) results in $\omega^{-2.5}$, whereas for $H_{\Delta\sigma} = H_\tau = 1/2$, Equation (22) indicates $\omega^{-2.75}$.

Table 2 summarizes the Hurst exponents with respect to various parameters: slip/stress drop (this article), slip velocity (ANDREWS, 1981) and stress on the fault (FRANKEL, 1991). These exponents are extended outside the range $0 < H < 1$ and aligned with the corresponding values: $H_u = H_{\Delta\sigma} + 1 = H_{\dot{u}} + 0.5 = H_\tau + 1$. For each category of Hurst exponent, the corresponding spectral amplitude falloffs of interest (in two dimension) as well as the ground displacement transforms are shown. The results from this study are consistent with ANDREWS' (1981) however, except in the case of the ω-square model, conflict with FRANKEL's (1991) for ground motion. Highlighted within the dashed lines in the table are those with Hurst exponents corresponding to $H_u = 1$, $H_{\Delta\sigma} = H_\tau = 0$ and $H_{\dot{u}} = 0.5$.

HERRERO and BERNARD's (1994) k-square model, which leads to the ω-square model, is consistent with the present model of $H_u = 1$. Their stress drop spectrum, which has a falloff of k^{-1} (where k is the radial wavenumber), agrees with the stress drop distribution with $H_{\Delta\sigma} = 0$ obtained here. Additionally, the relationship of

Table 2

Summary of the Hurst exponents with respect to various parameters and their corresponding spectral amplitude falloffs (in two dimension), together with the accompanied ground displacement transforms. Also shown here are the results of ANDREWS (1981) and FRANKEL (1991)

This study					ANDREWS (1981)			FRANKEL (1991)		
Slip (u)		Stress drop ($\Delta\sigma$)		$D(\omega)$	Slip velocity ($\dot{u}$)		$D(\omega)$	Stress (τ)		$D(\omega)$
H_u	$k^{-(H_u+1)}$	$H_{\Delta\sigma}$	$k^{-(H_{\Delta\sigma}+1)}$	$\omega^{-(H_{\Delta\sigma}+2)}$	$H_{\dot{u}}$	$k^{-(H_{\dot{u}}+1)}$	$\omega^{-(H_{\dot{u}}+1.5)}$	H_τ	$k^{-(H_\tau+1)}$	$\omega^{-(1.5H_\tau+2)}$
2.0	k^{-3}	1	k^{-2}	ω^{-3}	1.5	$k^{-2.5}$	ω^{-3}	1	k^{-2}	$\omega^{-3.5}$
1.5	$k^{-2.5}$	0.5	$k^{-1.5}$	$\omega^{-2.5}$	1	k^{-2}	$\omega^{-2.5}$	0.5	$k^{-1.5}$	$\omega^{-2.75}$
1	k^{-2}	0	k^{-1}	ω^{-2}	0.5	$k^{-1.5}$	ω^{-2}	0	k^{-1}	ω^{-2}
0.5	$k^{-1.5}$	-0.5	$k^{-0.5}$	$\omega^{-1.5}$	0	k^{-1}	$\omega^{-1.5}$	-0.5	$k^{-0.5}$	$\omega^{-1.25}$
0	k^{-1}	-1	k^{0}	ω^{-1}	-0.5	$k^{-0.5}$	ω^{-1}	-1	k^{0}	$\omega^{-0.5}$

H_X: Hurst exponent with respect to random variable X
u: slip
$\dot{u}$: slip velocity
$\Delta\sigma$: stress drop
τ: stress
k: wavenumber ($\sqrt{k_1^2 + k_3^2}$)
D: ground displacement in the frequency domain
ω: angular frequency

spectrum decays between the slip and the stress drop obtained by HERRERO and BERNARD (1994) is consistent with that presented in this article, which leads to $H_u = H_{\Delta\sigma} + 1$.

JOYNER (1991) adopted a source function (slip) that has a high-frequency falloff of $k^{-1.5}$ in the two-dimensional wavenumber space and a slip-velocity function of the Kostrov type (KOSTROV, 1964) with a high-frequency falloff of $\omega^{-0.5}$. The purpose of this was to provide ground motion with an ω-square displacement spectrum. In this paper, a Fourier spectrum of slip in the two-dimensional space with a high-frequency falloff of k^{-2} (see Table 2 with $H_u = 1$) is needed in order to obtain a similar ω-square ground motion displacement spectrum.

Discussion and Conclusions

From the perspective of fBm, the relationship that is reached in the present study,

$$H_{\Delta\sigma} = H_\tau = H_u - 1 = H_{\dot{u}} - 0.5, \tag{23}$$

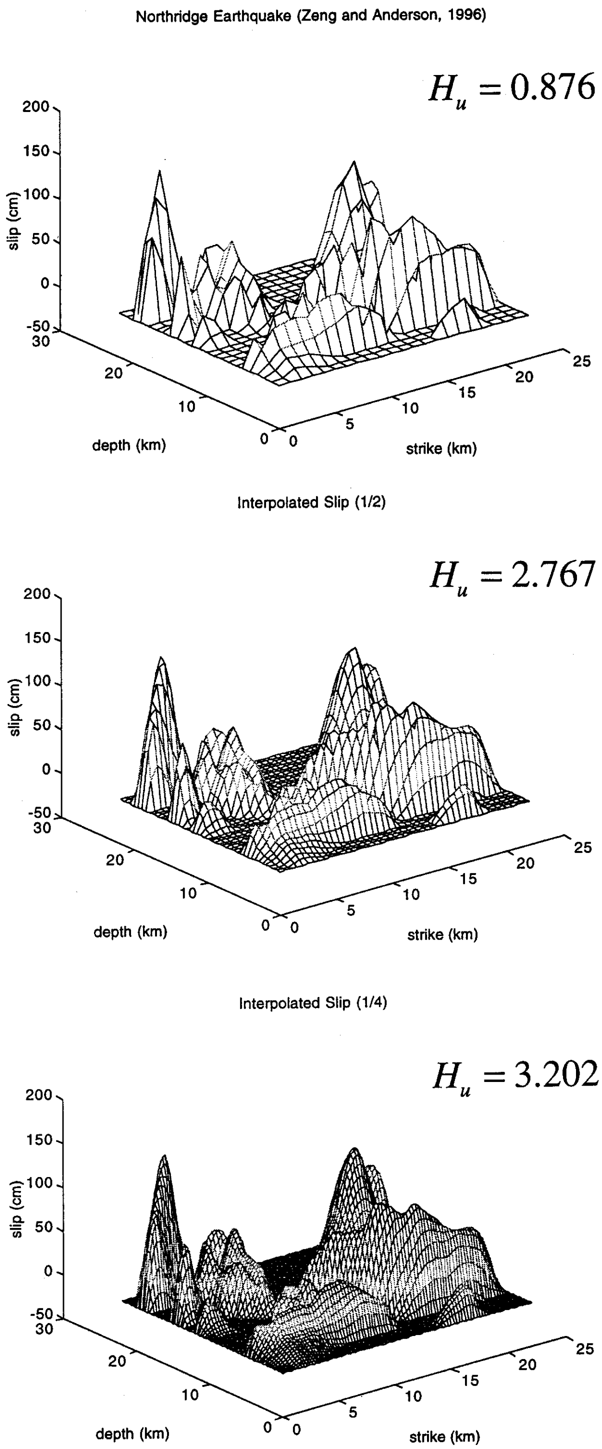
is consistent with the work of ANDREWS (1981). If fluctuations over one thousand bars of stress occur at shorter wavelengths (ANDREWS, 1981), the stress drop (or stress) becomes so highly erratic that a value of $H_{\Delta\sigma}$ (or H_τ) approaching zero can be deemed a reasonable postulation. However, it should never be less than or equal to zero (according to the theory of fBm) because a random process diverges for the Hurst exponent $H \leq 0$. That is, if $H_{\Delta\sigma} \leq 0$, the stress drop has increasing fluctuations over an area shrinking toward zero, which is physically unrealistic. As such, it appears reasonable that both $H_{\Delta\sigma}$ and H_τ are in the same range of 0 to 0.5, and preferably, are positive and close to 0.

Equation (23) (or more specifically, Equation (17)) implies that H_u is close to (but not less than) 1 if $H_{\Delta\sigma}$ is positive and close to 0. In such a case, the slip distribution should not be a self-similar (or more precisely, self-affine) process; on the contrary, it should be a process that is more likely differentiable. This is consistent with the inferred gross mean obtained by Equation (9) although is in conflict with those obtained by Equations (7) and (8) shown in Table 1. On the basis of the above inference, it can be stated that the author prefers the results of Equation (9) to those of Equations (7) and (8).

It seems that regardless of which method is used, the results shown in Table 1 depend upon slip distribution models. Although such a dependency is somewhat inevitable, possible reasons may be attributed to the error associated with each model and to the smoothing constraint included in the inversion implemented by various researchers. The total errors in the slip model contain contributions from both the inversion method and data errors. Some limits are posed in this study by the wavenumber bandwidth of the slip models. The bandwidth is, in general, limited at the high end by the Nyquist wavenumber associated with the element size of the grid as displayed in Table 1, compounded by the smoothing constraints. At the low end, the bandwidth is limited by lowcut filtering of the ground motion records and, for the purposes of this study, the necessity to use data with wavenumbers $k_x \gg k_{0x}$ and $k_y \gg k_{0y}$. For instance, the bandwidth along the strike direction that is used to estimate H_u is between 0.0333 (the corner wavenumber, in 1/km) and 1 (the Nyquist wavenumber) using Beroza and Spudich's slip model for the Morgan Hill earthquake with grid points 61 and element size 0.5 km. The corner wavenumber is calculated by $1/((61-1) \times 0.5)$, but the Nyquist wavenumber by $1/(2 \times 0.5)$.

It should be noted that any smoothing effects of the inversion tend to make the resulting H_u values larger than they should be because the smoothing constraint in the inversion minimizes the slip gradient or the slip curvature across the fault plane. Figure 3 shows an example of the smoothing effects on the resulting H_u values. Shown at the top of the figure is the original slip model of the Northridge earthquake from ZENG and ANDERSON (1996); the Hurst exponent of the slip for this model is 0.876 (the medium value obtained by Equation (9) as listed in Table 1). If the slip model is interpolated by making the dimension of the element size one-half of the original size, that is $0.5 \times 0.5\ \text{km}^2$ (as shown in the middle of the figure), the slip distribution is smoothed by the interpolation and the new H_u value is 2.767. If the slip model is further smoothed by making the element dimension one-fourth of the original size, that is $0.25 \times 0.25\ \text{km}^2$ (as shown at the bottom of the figure), the H_u value is 3.202. Analogously, for the slip model of the Landers earthquake from COHEE and BEROZA (1994), the H_u value changes from 0.624 to 2.262 and 2.477. Also, for the slip model of the Morgan Hill earthquake from BEROZA and SPUDICH (1988), the H_u value changes from 1.431 to 1.688 and 1.772. These examples demonstate that the smoothing effect on the resulting H_u values is tremendous, especially for those slip models whose Hurst exponents were originally estimated at less than 1.

If the smoothing effects are avoidable in the inversion, estimates of H_u may tend to be smaller than those presented in Table 1. Therefore, it is possible that the gross mean of H_u in Table 1 is less than 1, even if this is based on the method of Equation (9). A reasonable deduction would be that H_u is less than 1 in crustal earthquake phenomena only over some specific (or limited) bandwidth. Furthermore, since $H_{\Delta\sigma}$ should never be less than zero, some different physics may apply over this particular bandwidth such that the relation of Equation (17) may no longer be valid. The stress drop in the wavenumber domain may not be the multiplication of stiffness and slip as described in ANDREWS (1980). There remain aspects of the faulting process that cannot be described adequately by the relationship between ''static slip'' and ''static stress drop''. If this is the case, Equation (17) may not be valid to describe the ''dynamic'' phenomenon of the faulting process for a certain bandwidth of wavenumber. In such cases, multi-fractals should be used to explore the characteristics of the Hurst exponents of slip as long as the quality of


Northridge Earthquake (Zeng and Anderson, 1996)
$H_u = 0.876$
slip (cm)
200
150
100
50
0
-50
30
20
10
0
depth (km)
0
5
10
15
20
25
strike (km)
Interpolated Slip (1/2)
$H_u = 2.767$
slip (cm)
depth (km)
strike (km)
Interpolated Slip (1/4)
$H_u = 3.202$
slip (cm)
depth (km)
strike (km)

the slip data allows this to be done. That is, there should be more than one value of H_u in the slip distribution for different bandwidths of wavenumber. The range of the specific bandwidth may vary from one event to another, depending on the crustal environment that is involved.

Whether Equation (17) is unassailable or not, it can be stated at present that for the static description of faulting, $H_u \cong 1$ and $H_{\Delta\sigma} \cong 0$ appear to be good first approximations for slip and stress drop distributions. Thus, the spectrum of ground displacement that falls off as $\omega^{-(H_{\Delta\sigma}+2)}$ with $H_{\Delta\sigma}$ close to 0, implying a falloff of about ω^{-2} (but not flatter than ω^{-2}), is a good approximation for the purposes of ground motion simulations, and this is most consistent with strong-motion observations.

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Figure 3

Illustration of the smoothing effects on the H_u values. The original value of H_u for the Northridge earthquake (Zeng and Anderson's slip model) is 0.876 (top). When the slip model is interpolated with the element size of the grids reduced to one-half of the original, H_u is changed to 2.767 (middle). When the element size of the grids is reduced to one-fourth of the original, the new value of H_u is 3.202 (bottom).

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